Indian Statistical Institute
Semestral Examination 2008-2009
B.Math Hons.I year

Probability Theory I — 08-12-2008 — Max Marks 100

1. There are $n$ objects. An object is chosen at random and replaced. The experiment is repeated. Let $R$ be the number of trials required for one of the objects to be sampled twice. Note that $R$ takes values $2, 3, \cdots n+1$

   (a) Describe a sample space for the experiment. [5]

   (b) Calculate $q_r \equiv P(R = r), \qquad 2 \leq r \leq n+1.$ [6]

   (c) For $2 \leq r \leq n+1$, let $P_r$ be the probability that more than $r$ trials are required for an object to be sampled twice. Show that $P_r = \frac{(n)_r}{n^r}$ where $(n)_r \equiv n(n-1)\cdots(n-r+1)$. Hence deduce that $q_2 + \cdots + q_{n+1} = 1$.

   [9]

2. Let $n > 1$ and $-n \leq x \leq n$ be integers. A path from $(0,0)$ to $(n,x)$ is a sequence $\omega = (\omega_1, \cdots \omega_n)$, $\omega_i - \omega_{i-1} = \pm 1$, $i = 1 \cdots n$, $\omega_0 \equiv 0$ and $\omega_n = x$

   (a) For each $x, -n \leq x \leq n$, calculate the number of paths from $(0,0)$ to (n,x). [5]

   (b) Suppose all paths are equally likely. Let $E_{2n} = \{\omega : \omega_{2n} = 0, \}$ be all paths from $(0,0)$ to $(2n,0)$. Show that $\sqrt{\pi n}\, P(E_{2n}) \to 1$ as $n \to \infty$.

   [10]

3. Suppose events occur at random time points $T_i \in (0,\infty), i = 1, 2, \cdots$. Let $N_t :=$ number of events in $(0,t]$ for $t > 0, N_0 \equiv 0$. Suppose for disjoint intervals, $(0,t_1], (t_1,t_2], \cdots (t_{n-1},t_n]$ the random variables $N_{t_1}, N_{t_2} - N_{t_1}, \cdots N_{t_n} - N_{t_{n-1}}$ are independent. Under suitable assumptions show that $N_t$ is a Poisson random variable. [10]

4. Let $X \sim N(0,1)$.

   (a) Show that $\mathrm{E}X^{2k} = 1 \cdot 3 \cdot 5 \cdots (2k-1)$ for $k = 1, 2, \cdots$ (Hint: use integration by parts, successively, to calculate LHS.) [10]

(b) Show that for $s \geq 0$, the moment generating function of $X$ is given by $\mathrm{E}\, e^{sX} = e^{\frac{s^2}{2}}$. [10]

5. Let $X$ be a non negative random variable having a density. Show that

(a) $EX = \int_0^\infty P(X > x)dx$. [10]

(b) Deduce from a) that for $n \geq 1$,

$$EX^n = \int_0^\infty n x^{n-1} P(X > x)dx.$$

[10]

6. Suppose $X$ has density

$$f(x) = \frac{1}{\pi} \frac{1}{1+x^2} \qquad -\infty < x < \infty.$$

(a) Calculate the median of $X$. [8]

(b) Show that the random variable

$$Y = \begin{cases} 1/X & X \neq 0, \\ 0 & X = 0, \end{cases}$$

also has the density $f(x)$. [12]